Solution to Final Examination (A)

MAT1322-3X, Summer 2016

Part I. Multiple-choice Questions (3 × 10 = 30 marks)

CECAE DCDAB

1. The area of the region under the graph of $y = -2x^2 + 9$ and above the graph of $y = x^2 - 6x$ is

(A) 22; (B) 27; (C) 32; (D) 37; (E) 42.

Solution. (C) The intersections of these curves are found by the equation $-2x^2 + 9 = x^2 - 6x$, $3x^2 - 6x - 9 = 0$, or $x^2 - 2x - 3 = 0$. Then $x = -1$, $x = 3$.

$$A = \int_{-1}^{3} (-2x^2 + 9 - x^2 + 6x)dx = \int_{-1}^{3} (-3x^2 + 6x + 9)dx = \left[-x^3 + 3x^2 + 9x\right]_{x=-1}^{3} = 32$$

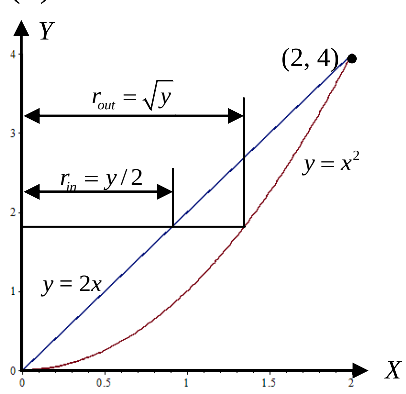
2. Let R be the region above the parabola $y = x^2$ and under the line $y = 2x$. Solid B is obtained by revolving R **about the *y*-axis**. Then the volume of B is calculated by the integral

(A) $\pi\int_0^2 ((2x)^2 - (x^2)^2)dx$; (B) $\pi\int_0^2 \left(\left(\frac{y}{2}\right)^2 - (\sqrt{y})^2\right)dy$;

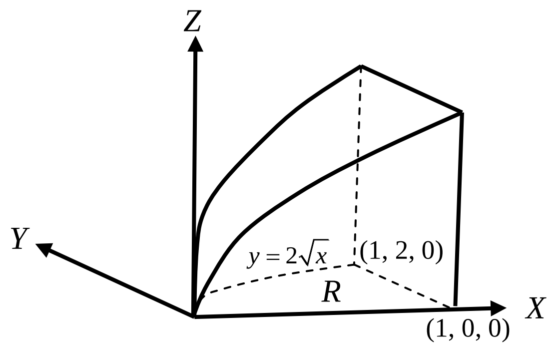
(C) $\pi\int_0^4 \left(\left(\frac{y}{2}\right)^2 - (\sqrt{y})^2\right)dy$; (D) $\pi\int_0^2 \left((\sqrt{y})^2 - \left(\frac{y}{2}\right)^2\right)dy$;

(E) $\pi\int_0^4 \left((\sqrt{y})^2 - \left(\frac{y}{2}\right)^2\right)dy$.

Answer. (E)



3. Let R be the region between the graph $y = 2\sqrt{x}$ and the x-axis, $0 \le x \le 1$. A solid has R as its base, and the cross sections perpendicular to the x-axis are squares. The solid is shown in the following figure:

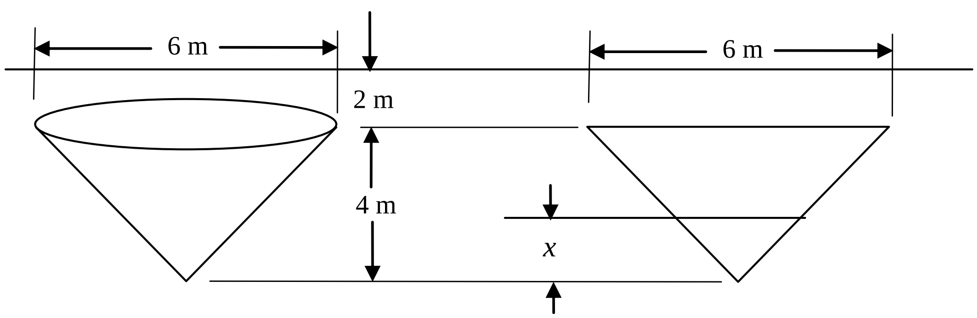


The volume of the solid is

(A) 1; (B) $\frac{3}{2}$; (C) 2; (D) $\frac{5}{2}$; (E) 3.

Solution. (C) The area of a cross section at a given value x is $A(x) = (2\sqrt{x})^2 = 4x$. The volume of the solid is $V = 4\int_0^1 x\,dx = 2\left[x^2\right]_{x=0}^1 = 2$.

4. A container has the shape of an inverted circular cone as in the following figure:



Suppose the container is filled with water of density ρ kg / m^3. Let x be the distance between a level of water and the bottom of the container, and let g be the acceleration of gravity. Then the work, in Joules, needed to pump the water in the container to a point 2 meters above the top of the container is calculated by the integral

(A) $\frac{9\rho g\pi}{16}\int_0^4 x^2(6-x)dx$; (B) $\frac{9\rho g\pi}{16}\int_0^6 x^2(6-x)dx$; (C) $\frac{9\rho g\pi}{16}\int_0^4 x^2(x+2)dx$;
(D) $\frac{9\rho g\pi}{16}\int_0^4 x^2(4-x)dx$; (E) $\frac{9\rho g\pi}{16}\int_0^6 x^2(4-x)dx$.

Solution. (A) Consider a layer of water with distance x from the bottom. The thickness of this layer is dx. The diameter of this layer is $6x / 4 = 3x / 2$. The radius is $3x / 4$. The area of the layer is $(3x / 4)^2\pi$. The weight of this layer is $\rho g(3x / 4)^2\pi\,dx$. The work needed to pump this layer to a point 2 meters above the top of the container is $dW = \rho g(3x/4)^2(6-x)\pi\, dx$. The bottom layer has $x = 0$, and the top layer has $x = 4$. The total work needed is $W = \dfrac{9\rho g\pi}{16}\int_0^4 x^2(6-x)dx$.

5. Consider improper integral $\int_1^\infty \frac{2\sqrt{x}-1}{x+\sqrt{x}}dx$. Which one of the following argument is true?

(A) When $x > 1$, $\frac{2\sqrt{x}-1}{x+\sqrt{x}} > \frac{2\sqrt{x}}{x} = \frac{2}{\sqrt{x}}$. Since $\int_1^\infty \frac{2}{\sqrt{x}}dx$ diverges, $\int_1^\infty \frac{2\sqrt{x}-1}{x+\sqrt{x}}dx$ diverges.

(B) When $x > 1$, $\frac{2\sqrt{x}-1}{x+\sqrt{x}} < \frac{2\sqrt{x}}{x} = \frac{2}{\sqrt{x}}$. Since $\int_1^\infty \frac{2}{\sqrt{x}}dx$ diverges, $\int_1^\infty \frac{2\sqrt{x}-1}{x+\sqrt{x}}dx$ diverges.

(C) When $x > 1$, $\frac{2\sqrt{x}-1}{x+\sqrt{x}} < \frac{2\sqrt{x}}{x} = \frac{2}{\sqrt{x}}$. Since $\int_1^\infty \frac{2}{\sqrt{x}}dx$ converges, $\int_1^\infty \frac{2\sqrt{x}-1}{x+\sqrt{x}}dx$ converges.

(D) When $x > 1$, $\frac{2\sqrt{x}-1}{x+\sqrt{x}} > \frac{\sqrt{x}}{2x} = \frac{1}{2\sqrt{x}}$. Since $\int_1^\infty \frac{1}{2\sqrt{x}}dx$ converges, $\int_1^\infty \frac{2\sqrt{x}-1}{x+\sqrt{x}}dx$ converges.

(E) When $x > 1$, $\frac{2\sqrt{x}-1}{x+\sqrt{x}} > \frac{\sqrt{x}}{2x} = \frac{1}{2\sqrt{x}}$. Since $\int_1^\infty \frac{1}{2\sqrt{x}}dx$ diverges, $\int_1^\infty \frac{2\sqrt{x}-1}{x+\sqrt{x}}dx$ diverges.

Solution. (E) (A) is false because $\frac{2\sqrt{x}-1}{x+\sqrt{x}} < \frac{2\sqrt{x}}{x} = \frac{2}{\sqrt{x}}$. (B) is false because the integral of the bigger function diverges does not imply that the integral of the smaller function diverges. (C) is false because $\int_1^\infty \frac{1}{2\sqrt{x}}dx$ diverges. (D) is false because the integral of the small function converges does not imply that the integral of the bigger function converges.

6. If $y = f(t)$ is the solution of the initial-value problem $y' = 4y - y^2$, $y(0) = 1$. Which one of the following values is closest to the value $y(1)$?

(A) 3.5; (B) 3.6; (C) 3.7; (D) 3.8; (E) 3.9.

Solution. (D) Separating variables, $\int \frac{dy}{y(4-y)} = \int dt$. Hence, $\int \frac{dy}{y(4-y)} = \frac{1}{4}\int \left(\frac{1}{y} + \frac{1}{4-y}\right)dy = \frac{1}{4}\ln\left|\frac{y}{4-y}\right| = t + C$. Then $\frac{y}{4-y} = Ke^{4t}$, where $K = \pm e^{4C}$. By the initial condition, $K = \frac{1}{3}$. $y = \frac{(4/3)e^{4t}}{1+(1/3)e^{4t}} = \frac{4}{1+3e^{-4t}}$. $y(1) = \frac{4}{1+3e^{-4}} \approx 3.792$.

7. Suppose Euler's method with step size $h = 0.1$ is used to find an approximation of $y(0.3)$, where y is the solution to the initial-value problem $y' = (2t - 1)y^2$, $y(0) = 1$. Which one of the following is closest to the answer?

(A) 0.900; (B) 0.835; (C) 0.793; (D) 0.768; (E) 0.756.

Solution. (C)

i	t_i	y_i
0	0	1
1	0.1	$1 + 0.1 \times (2 \times 0 - 1) \times 1^2 = 0.9$
2	0.2	$0.9 + 0.1 \times (2 \times 0.1 - 1) \times 0.9^2 \approx 0.8352$
3	0.3	$0.8352 + 0.1 \times (2 \times 0.2 - 1) \times 0.8352^2 \approx 0.7933$

8. Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{5^n n}(x+2)^n$. Which one of the following statement is true?

(A) This series absolutely converge when $-7 < x < 3$; it conditionally converges at $x = -7$; it diverges when $x < -7$ or $x \geq 3$.
(B) This series absolutely converges when $x = -7$; it conditionally converges when $-7 < x < 3$; it diverges when $x < -7$ or when $x \geq 3$.
(C) This series absolutely converges when $-3 < x < 7$; it conditionally converges at $x = -3$; it diverge when $x < -3$ or $x \geq 7$.
(D) This series absolutely converges when $-7 < x < 3$; it conditionally converge at $x = 3$; it diverges when $x \leq -7$ or $x > 3$.
(E) This series is absolutely converges when $x = -3$; it conditionally converges when $-3 < x < 7$; it diverges when $x < -3$ or $x \geq 7$.

Solution. (D) The center of the series is -2. The radius of convergence is $\lim_{n\to\infty} \frac{5^{n+1}(n+2)}{5^n n} = 5$. Then this series is absolutely convergent when $-2 - 5 < x < -2 + 5$, or $-7 < x < 3$. When $x = -7$, the series become $\sum_{n=1}^{\infty} \frac{(-1)^n}{5^n n}(-5)^n = \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. When $x = 3$, this series become $\sum_{n=1}^{\infty} \frac{(-1)^n}{5^n n} 5^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which converges by the alternating series test. Therefore, this series is conditionally convergent at $x = 3$, and divergent when $x \leq -7$ or $x > 3$.

9. If a function $z = f(x, y)$ is defined implicitly by the equation $x^2z^2 + 3xy + yz^3 = 3$, then the equation of the tangent plane at the point $(2, -1, 3)$ is

(A) $z = 11x + 11y - 8$; (B) $z = 7x + 8y - 3$; (C) $z = 11x + 13y - 6$;
(D) $z = 13x + 11y - 12$; (E) $z = 8x + 7y - 6$.

Solution. (A) Let $F(x, y, z) = x^2z^2 + 3xy + yz^3 - 3$. $F_x = 2xz^2 + 3y$, $F_y = 3x + z^3$, $F_z = 2x^2z + 3yz^2$.

At point $(2, -1, 3)$, $F_x = 33$, $F_y = 33$, $F_z = -3$. The equation of the tangent plane is $33(x-2) + 33(y+1) - 3(z-3) = 0$, or $33x + 33y - 3z = 24$, i.e., $z = 11x + 11y - 8$.

10. The directional derivative of the function $z = 2x^2y - y^3$ at the point $x = 2$ and $y = -1$ in the direction of the vector $\mathbf{u} = (3, 4)$ is

(A) $\frac{1}{5}$; (B) $-\frac{4}{5}$; (C) $-\frac{1}{5}$; (D) $\frac{2}{5}$; (E) $-\frac{34}{5}$.

Solution. (B) The unit vector in the direction of $\mathbf{u}$ is $\mathbf{v} = \left(\frac{3}{5}, \frac{4}{5}\right)$. $z_x = 4xy$, $z_y = 2x^2 - 3y^2$. When $x = 2$ and $y = -1$, $z_x(2, -1) = -8$, $z_y(2, -1) = 5$. $D_{\mathbf{u}}(z) = -8\cdot\frac{3}{5} + 5\cdot\frac{4}{5} = \frac{-24+20}{5} = -\frac{4}{5}$.

Part II. Long Answer Questions (20 marks)

1. (3 marks) Use **the definition of the improper integral** to find the value of $\int_1^2 \frac{1}{\sqrt{x-1}}\,dx$.

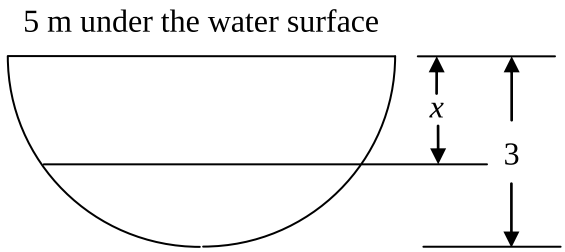
Solution. $\int_1^2 \frac{1}{\sqrt{x-1}}\,dx = \lim_{a\to 1^+}\int_a^2 \frac{1}{\sqrt{x-1}}dx = \lim_{a\to 1^+}\int_{a-1}^1 \frac{1}{\sqrt{u}}du = \lim_{a\to 1^+}\left[2\sqrt{u}\right]_{u=a-1}^1 = \lim_{a\to 1^+}(2 - 2\sqrt{a-1}) = 2$.

2. (3 marks) Suppose salted water of concentration 0.5 g / m^3 is added to a pool of volume 3000 m^3 at a rate 12 m^3/ minute. The water in the pool is well mixed and the same amount of water (i.e., 12 m^3 / minute) overflows from the pool. Let $Q(t)$ be the quantity, in grams, of salt in the pool at time t. Find a differential equation that is satisfied by function $Q(t)$. (You don't need to solve this equation!)

Solution. $rate_{\text{in}} = 0.5 \times 12 = 6$ g / min. $rate_{\text{out}} = \frac{Q}{3000}\times 12 = 0.004$ g / min.

The equation is $Q' = 6 - 0.004Q$.

3. (4 marks) Suppose the lower half a circular disk of radius 3 meters is submerged vertically into water (with density ρ kg/m^3) so that the top of the half disk is 5 meters under the water surface. Construct (but not evaluate) a definite integral that can be used to calculate the force (in Newtons) acting on this surface. Make sure you define an appropriate variable, and show step by step how the integral is constructed. Write clearly what is the meaning of the value that you calculate in each step. Make clear the meaning of every symbol that you use in your solution.
Your presentation is a part of the marking scheme on top of a correct answer!



Solution. There are a number of different ways to define the variable to specify a horizontal slice of the surface. The following is one of the cases:

Let x be the distance between a horizontal slice of the half disk and the top of the half disk. The thickness of the slice is dx.

Then area of the slice is $A(x) = 2\sqrt{9-x^2}\,dx$, and the depth of the slice is $x + 5$.

The force acting on this slice is $dF = 2\rho g(x + 5)\sqrt{9-x^2}$.

Since the top of the half disk has $x = 0$ and the bottom of the disk has $x = 3$, the total force acting on this half disk is calculated by

$$F = 2\rho g\int_0^3 (x+5)\sqrt{9-x^2}\,dx.$$

Alternative solutions:

A. Let x be the **depth** of a horizontal slice with thickness dx.

Then area of the slice is $A(x) = 2\sqrt{9-(x-5)^2}\,dx$, and the depth of the slice is x.

The force acting on this slice is $dF = 2\rho gx\sqrt{9-(x-5)^2}$.

Since the top of the half disk has $x = 5$ and the bottom of the disk has $x = 8$, the total force acting on this half disk is calculated by

$$F = 2\rho g\int_5^8 x\sqrt{9-(x-5)^2}\,dx.$$

B. Let x be the distance between the bottom level of the half disk and a horizontal slice with thickness dx.

Then area of the slice is $A(x) = 2\sqrt{9-(3-x)^2}\,dx$, and the depth of the slice is $8 - x$.

The force acting on this slice is $dF = 2\rho g(8-x)\sqrt{9-(3-x)^2}$.

Since the top of the half disk has $x = 3$ and the bottom of the disk has $x = 0$, the total force acting on this half disk is calculated by

$$F = 2\rho g\int_0^3 (8-x)\sqrt{9-(3-x)^2}\,dx .$$

4. (6 marks) Use an appropriate test method to determine whether each of the following series is convergent or divergent. State the conditions why the test method applies.

(a) $\sum_{n=1}^{\infty}\frac{3^n}{5^n-2^n}$; (b) $\sum_{n=0}^{\infty}(-1)^n\sqrt{\frac{n}{n+1}}$.

Solution. (a) This question can be solved in a number of different ways.

(i) Use comparison test:

Since $\frac{3^n}{5^n-2^n} < \frac{3^n}{5^n/2} < \frac{3^{n+1}}{5^n}$, and $\sum_{n=0}^{\infty}\frac{3^{n+1}}{5^n}$ is a geometric series with common ratio $r = \frac{3}{5}$, which converges. Hence, this series converges.

(ii) Use limit comparison test:

Let $a_n = \frac{3^n}{5^n-2^n}$ and let $b_n = \frac{3^n}{5^n}$. Then $\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{3^n}{5^n-2^n}\times\frac{5^n}{3^n} = \lim_{n\to\infty}\frac{1}{1-(2/5)^n} = 1$. Since $\sum_{n=0}^{\infty}b_n = \sum_{n=0}^{\infty}\frac{3^n}{5^n}$ converges as a geometric series with common ratio $r = \frac{3}{5}$, this series converges.

(iii) Use the ratio test:

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{3^{n+1}}{5^{n+1}-2^{n+1}}\times\frac{5^n-2^n}{3^n} = 3\lim_{n\to\infty}\frac{5^n-2^n}{5^{n+1}-2^{n+1}} = 3\lim_{n\to\infty}\frac{1-(2/5)^n}{5-2(2/5)^n} = \frac{3}{5} < 1$. Hence, this series converges.

Since this series is a positive series, the comparison test, limit comparison test and ratio test works.

(b) Since $\lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = 1$, the general term of this series does not approach 0. This series diverges.

5. (4 marks) The Maclaurin series of the function $y = \sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} .$$

Find the first four non-zero terms of the Maclaurin series of the function $F(x) = \int_0^x \cos(2t^2)dt$.

Solution. Taking the derivative of the sine function, we have

$$\cos x = (\sin x)' = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots .$$

With $x = 2t^2$,

$$\cos (2t^2) = 1 - \frac{(2t^2)^2}{2!} + \frac{(2t^2)^4}{4!} - \frac{(2t^2)^6}{6!} + \dots = 1 - 2t^4 + \frac{2}{3}t^8 - \frac{4}{45}t^{12} + \dots .$$

Therefore,

$$F(x) = \int_0^x \cos(2t^2)dt = x - \frac{2x^5}{5} + \frac{2x^9}{9\times 3} - \frac{4x^{13}}{13\times 45} + \dots = x - \frac{2x^5}{5} + \frac{2x^9}{27} - \frac{4x^{13}}{585} + \dots .$$